

## DIFFERENT DISTRIBUTIONS IN QUEUEING NETWORKS

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### ABSTRACT

In this chapter we will derive the equilibrium distribution of states of a model containing four different types of service centers and  $R$  different classes of customers. From this steady state distribution one can compute the moments of the queue sizes for different classes of customers at different service centers, the utilizations of the service centers, the “cycle time” or response time for different classes of customers, the “throughput” of different classes of customers, and other measures of system performance.

**KEYWORDS:** Queueing System, Equilibrium State Probabilities, Poisson Process and the Exponential Distribution

### INTRODUCTION

Chandy [4] defines another type of balance equation which he calls the local balance equations. Informally, a local balance equation equates the rate of flow into a state by a customer entering a stage of service to the flow out of that state due to a customer leaving that stage of service. We associate a customer with a stage of service in the following ways. If the customer is in service at a service center, then he is in one of the stages of his service time distribution at that service center. If the customer is queued at a service center, then he is in the stage of his service time distribution he will enter when next given service.

To illustrate the concept of local balance we consider the relatively simple network model in Figure 1.

This is a closed network with two classes of customers (which we refer to as class 1 and class 2). There are  $N_1$  class 1 customers and  $N_2$  class 2 customers in the networks. All service times are exponentially distributed and

$\frac{1}{\mu_{ir}} (i = 1, 2, r = 1, 2)$  is the mean service time for a class  $r$  customer at service center  $i$ .

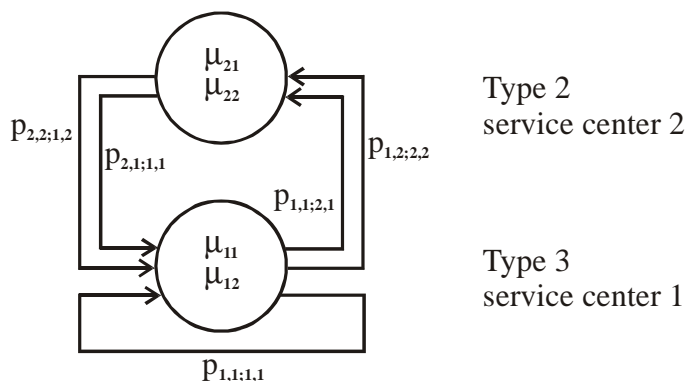


Figure 1: In this Example  $P_{1,2;2,2} = P_{2,2;1,2} = P_{2,1;1,1} = 1, P_{1,1;1,1} + P_{1,1;2,1} = 1$

Let  $n_{i,r}$  be the number of class  $r$  customers at service center  $i$ . For convenience we write the global and local balance equations only for the states in which  $n_{ir} > 0$ ,  $i = 1, 2$ ,  $r = 1, 2$ .

Global Balance Equation:

$$\begin{aligned}
 & P(n_{11} - 1, n_{12}, n_{21} + 1, n_{22}) \left( \frac{n_{21} + 1}{n_{21} + n_{22} + 1} \right) \mu_{21} \\
 & + P(n_{11} + 1, n_{12}, n_{21} - 1, n_{22}) (n_{11} + 1) \mu_{11} P_{1,1;2,1} \\
 & + P(n_{11}, n_{12}, n_{21}, n_{22}) n_{11} \mu_{11} P_{1,1;1,1} + P(n_{11}, n_{12} + 1, n_{21}, n_{22} - 1) (n_{12} + 1) \mu_{12} \\
 & + P(n_{11}, n_{12} - 1, n_{21}, n_{22} + 1) \left( \frac{n_{22} + 1}{n_{21} + n_{22} + 1} \right) \mu_{22}
 \end{aligned}$$

Local

Balance

Equations:

$$= P(n_{11}, n_{12}, n_{21}, n_{22}) \left[ n_{11} \mu_{11} + n_{12} \mu_{12} + \frac{n_{21}}{n_{21} + n_{22}} \mu_{21} + \frac{n_{22}}{n_{21} + n_{22}} \mu_{22} \right]$$

$$P(n_{11} - 1, n_{12}, n_{21} + 1, n_{22}) \left( \frac{n_{21} + 1}{n_{21} + n_{22} + 1} \right) \mu_{21} \quad (5.1)$$

$$+ P(n_{11}, n_{12}, n_{21}, n_{22}) n_{11} \mu_{11} P_{1,1;1,1} = P(n_{11}, n_{12}, n_{21}, n_{22}) n_{11} \mu_{11}$$

$$P(n_{11}, n_{12} - 1, n_{21}, n_{22} + 1) \left( \frac{n_{22} + 1}{n_{21} + n_{22} + 1} \right) \mu_{22} = P(n_{11}, n_{12}, n_{21}, n_{22}) n_{12} \mu_{12} \quad (5.2)$$

$$P(n_{11} + 1, n_{12}, n_{21} - 1, n_{22}) (n_{11} + 1) \mu_{11} P_{1,1;2,1} = \quad (5.3)$$

$$P(n_{11}, n_{12}, n_{21}, n_{22}) \left( \frac{n_{21}}{n_{21} + n_{22}} \right) \mu_{21}$$

$$P(n_{11}, n_{12} + 1, n_{21}, n_{22} - 1) (n_{12} + 1) \mu_{12} = P(n_{11}, n_{12}, n_{21}, n_{22}) \left( \frac{n_{22}}{n_{21} + n_{22}} \right) \mu_{22} \quad (5.4)$$

Since all the service time distributions in this, example are exponential the current stage of service of a customer is uniquely defined by the customer's class and the current service center. Local balance equation  $(i, r)$  for  $i = 1, 2$ ,  $r = 1, 2$  equates the rate of flow out of state  $(n_{11}, n_{12}, n_{21}, n_{22})$  due to a class  $r$  customer leaving service

Center  $i$  with the rate of flow into state  $(n_{11}, n_{12}, n_{21}, n_{22})$  due to a class  $r$  customer entering service center  $i$ .

The value of the local balance concept is that

- It leads to a simpler and more organized search for solutions for equilibrium state probabilities and
- It works for a large number of cases (in fact for virtually all of the closed form solutions known for general classes of networks of queues--although not many interesting cases have known solutions).

Before presenting the solution to the class of networks described, we define a set of terms that appear in the solution.

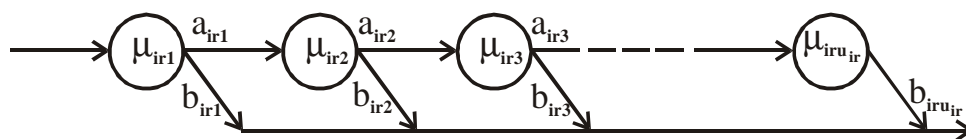
For a customer of class  $r$ , let  $\{e_{ir}, 1 \leq i \leq N\}$  be a solution to the following set of equations.

$$\sum_{1 \leq i \leq N} e_{ir} P_{i,r;j,s} + d_{js} = e_{js}, \quad 1 \leq j \leq N$$

The value of  $d_{js}$  is determined by the arrival process of customers of class  $s$  to service center  $j$ . If there are no such arrivals from outside the system, then  $d_{js} = 0$ . If there are such arrivals then  $d_{js} = q_{js}$ . In a closed system there are no arrivals to any center and all the  $d_{js}$  are zero. In this case  $e_{ir}$  is the relative frequency of visits to service center  $i$  by customers of class  $r$ .

Note that a system may be “open” with respect to some classes of customers and “closed” with respect to other classes of customers. Our solution applies to this class of system.

One further definition is required. If at the  $i^{\text{th}}$  service center the  $r^{\text{th}}$  class of customers has a service time distribution that is represented as a network of stages then this is represented as illustrated in figure 2.



**Figure 2: The First Subscript on  $a, b$  and  $\mu$  Denotes the Service Center**

The Second Subscript Denotes the Class of Customer And the Third Subscript Denotes the Stage.

$$\text{Let } A_{ir\ell} = \prod_{j=1}^{\ell} a_{irj}$$

### Theorem 1

Given a network of service stations which is open, closed or mixed in which each service center is of type 1, 2, 3 or 4. Then the equilibrium state probabilities are given by

$$P(S = x_1, x_2, \dots, x_N) = Cd(S) f_1(x_1) f_2(x_2) \dots f_N(x_N)$$

Where  $C$  is a normalizing constant chosen to make the equilibrium state probabilities sum to 1,  $d(S)$  is a function of the total number of customers in system and each  $f_i$  is a function that depends on the type of service center  $i$ .

If service center  $i$  is of type 1 then

$$f_i(x_i) = \prod_{j=1}^{n_i} \left[ \frac{1}{\mu_i(j)} e^{ix_{ij}} \right]$$

If service center  $i$  is of type 2 then

$$f_i(x_i) = n_i! \prod_{r=1}^R \prod_{\ell=1}^{u_{ir}} \left\{ \left[ \frac{e_{ir} A_{ir\ell}}{u_{ir\ell}} \right]^{m_{ir\ell}} \frac{1}{m_{ir\ell}!} \right\}$$

If service center  $i$  is of type 3 then

$$f_i(x_i) = \prod_{r=1}^R \prod_{\ell=1}^{u_{ir}} \left\{ \left[ \frac{e_{ir} A_{ir\ell}}{u_{ir\ell}} \right]^{m_{ir\ell}} \frac{1}{m_{ir\ell}!} \right\}$$

If service center  $i$  is of type 4 then

$$f_i(x_i) = \prod_{j=1}^{n_i} \left[ e_{ir_j} A_{ir_j} \frac{1}{m_j \mu_{ir_j m_j}} \right]$$

If the arrivals to the system depend on the total number of customers in the system,  $M(S)$ , and the arrivals are of class  $r$  and for center  $i$  according to fixed probabilities  $P_{ir}$  then

$$d(S) = \prod_{i=0}^{M(S)-1} \lambda(i)$$

If we have the second type of state dependent arrival process then

$$d(S) = \prod_{j=1}^m \prod_{i=0}^{M(E_j)-1} \lambda_j(i)$$

If the network is closed then  $d(S) = 1$ .

The theorem is proved by checking that the local balance equations are satisfied. In every case for which these results apply the local balance equations reduce to the defining equations for the  $\{e_{ir}\}$ .

## SIMPLIFICATION OF RESULTS

The solution presented for the equilibrium state probabilities deals with system states that are more detailed than

is usually required. The more detailed states are necessary to derive the equilibrium state probabilities. Now we define the system state as the number of each class of customer in each service center. More formally state  $S$  of the system is given by  $(y_1, y_2, \dots, y_N)$  where  $y_i = (n_{i1}, n_{i2}, \dots, n_{ir})$  and  $n_{ir}$  is the number of customers of class  $r$  in service center  $i$ . Let  $n_i$  be the total number of customers at service center  $i$  and let  $\frac{1}{\mu_{ir}}$  be the mean service time of a class  $r$  customer at service center  $i$ . Then the equilibrium state probabilities are given by

$$P(S = (y_1, y_2, \dots, y_N)) = Cd(S) g_1(y_1) g_2(y_2) \dots g_N(y_N)$$

Where

If service center  $i$  is of type 1 then

$$g_i(y_i) = n_i! \left\{ \prod_{r=1}^R \frac{1}{n_{ir}!} [e_{ir}]^{n_{ir}} \right\} \prod_{j=1}^{n_i} \frac{1}{\mu_i(j)}$$

if service center  $i$  is of type 2 or 4 then

$$g_i(y_i) = n_i! \prod_{r=1}^R \frac{1}{n_{ir}!} \left[ \frac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}}$$

If service center  $i$  is of type 3 then

$$g_i(y_i) = \prod_{r=1}^R \frac{1}{n_{ir}!} \left[ \frac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}}$$

In each case the expression for  $g_i(y_i)$  is derived by summing  $f_i(x_i)$  over all  $x_i$  with  $n_{i1}, n_{i2}, \dots, n_{ik}$  fixed. That this is the correct definition of the  $g_i$  follows from the product form of the solution given

The evaluation of the normalizing constant requires summing the given expression for the equilibrium state probabilities over all feasible states. In the next section we show a closed form solution for  $C$  for an open network in which  $\mu_i(m) = \mu_i$  for all  $m$  if service center  $i$  is of type one.

### Open Systems

For open systems it is possible to obtain a closed form solution for the normalization constant when the arrival process is of the first type and  $\lambda(M(S)) = \lambda = \text{constant}$ . Since the system is open any number of customers is feasible at a service center.

Therefore

$$c^{-1} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \left( \prod_{i=1}^N \lambda^{n_i} h_i(n_i) \right)$$

$$\text{Or } c^{-1} = \left( \sum_{n_1=0}^{\infty} \lambda^{n_1} h_1(n_1) \right) \left( \sum_{n_2=0}^{\infty} \lambda^{n_2} h_2(n_2) \right) \dots \left( \sum_{n_N=0}^{\infty} \lambda^{n_N} h_N(n_N) \right)$$

Also,

$$\sum_{n_i=0}^{\infty} h_i(n_i) = \left( 1 - \sum_{r \in R_i} \lambda \frac{e_{ir}}{\mu_i} \right)^{-1} \quad \text{If service center } i \text{ is type 1 and } \mu_i(n_i) = \mu_i$$

$$= \left( 1 - \sum_{r \in R_i} \lambda \frac{e_{ir}}{\mu_{ir}} \right)^{-1} \quad \text{If service center } i \text{ is type 2 or 4}$$

$$e^{\sum_{r \in R_i} \lambda \frac{e_{ir}}{\mu_{ir}}} \quad \text{If service center } i \text{ is type 3}$$

### Marginal Distribution at a Service Center in an Open System

Let  $P_i(n_i)$  be the equilibrium probability that there are  $n_i$  customers at service center  $i$ .

$$P_i(n_i) = C \lambda^{n_i} h_i(n_i) \prod_{\substack{j=1 \\ j \neq i}}^N \left( \sum_{n_j=0}^{\infty} \lambda^{n_j} h_j(n_j) \right)$$

Using the expression for  $C$ , we reduce this to

$$P_i(n_i) = \frac{\lambda^{n_i} h_i(n_i)}{\sum_{m=0}^{\infty} \lambda^m h_i(m)}$$

$$\text{Let } \rho_i = \sum_{r \in R_i} \lambda \frac{e_{ir}}{\mu_i} \quad \text{if service center } i \text{ is type 1.}$$

$$\rho_i = \sum_{r \in R_i} \lambda \frac{e_{ir}}{\mu_{ir}} \quad \text{if service center } i \text{ is type 2, 3 or 4.}$$

$$\text{Then } P_i(n_i) = (1 - \rho_i) \rho_i^{n_i} \quad \text{if service center } i \text{ is type 1, 2 or 4.}$$

$$= e^{-\rho_i} \frac{\rho_i^{n_i}}{n_i!} \quad \text{If service center } i \text{ is type 3.}$$

These results provide a convenient way of examining the equilibrium distribution at a service center, For type 1, 2 or 4 service stations the marginal distribution is the same as the distribution of the number of customers in an  $M / M / 1$  queue with a suitably chosen utilization,  $\rho_i$ . For the equilibrium solution to exist each  $\rho_i$  is required to be less than 1.

The marginal distribution for a type 3 service center is the same as the equilibrium distribution for the number of customers for an  $M / G / \infty$  system with  $\rho_i = \frac{\lambda}{\mu}$ . This certainly appears to be reasonable since for open properties of network models that satisfy local balance.

Consider a general service time distribution represented by a network of stages as in Figure 1. Let this represent the service time distribution for a customer in class  $r$  in service center  $i$ . We introduce  $n$  new customer classes denoted by  $r_1, r_2, \dots, r_n$  which correspond to the stages in this network and delete customer class  $r$ .

The service time of a class  $r_\ell$  customer will be exponential with mean  $\frac{1}{\mu_\ell} (1 \leq \ell \leq n)$ . The transition probabilities for a class  $r_\ell$  customer are defined as:

$$p_{i, r_\ell; j, s} = \ell P_{i, r; j, s}$$

$$p_{i, r_\ell; i, r_{\ell+1}} = a_\ell \quad 1 \leq \ell < n$$

To take care of the transitions into class  $r$  in the original model we require that all transitions into state  $r$  be redefined as transitions into state  $r_1$ . These transition probabilities are defined as:

$$P_{j, s; i, r_1} = P_{j, s; i, r}, \quad j, s$$

With this transformation of the model a customer will have the same distribution of total time at a service center and will have the same transition probabilities from service center to service center.

After performing this transformation for each customer class with a general service time distribution we have a model in which all service time distributions are exponential. Suppose that  $\frac{1}{\mu_{i, r}}$  is the mean service time of a class  $r$  customer at service center  $i$ . Let

$$\mu_i = \max_r [M_{i, r}]$$

We redefine the mean service time for each class of customers at service center  $i$  to be  $\frac{1}{\mu_i}$ . Now we redefine the transition probabilities out of service center  $i$ ,

$$\text{Let } P_r = 1 - \frac{\mu_{i,r}}{\mu_i}$$

Then define

$$P'_{i,r;i,r} = P_r + (1 - p_r) P_{i,r;i,r}$$

$$P'_{i,r;j,s} = (1 - p_r) P_{i,r;j,s}$$

The effect of these new transition probabilities is to cause a class  $r$  customer to be fed back (or to revisit) service center  $i$  a random number of times. Each time the class  $r$  customer enters service center  $i$  his service time is exponentially distributed with mean  $\frac{1}{\mu_i}$ . The number of visits the class  $r$  customer makes to service center  $i$  (between transitions in the

original model) is geometrically distributed with mean  $\frac{1}{1 - p_r}$ . It is easily shown that the total service time of the class  $r$

customer at service center  $i$  is exponentially distributed with mean  $\left( \frac{1}{1 - p_r} \right) \frac{1}{\mu_i} = \frac{1}{\mu_{i,r}}$  by Feller [8]. Therefore we

have not changed the total service time distribution for this customer at service center  $i$ .

### Poisson Process and the Exponential Distribution

An arrival process  $\{N(t), t \geq 0\}$ , where  $N(t)$  denotes the total number of arrivals up to  $t$ , with  $N(0) = 0$ , and which satisfies the following three assumptions:

- The probability that an arrival occurs between time  $t$  and time  $t + \Delta t$  is equal to  $\lambda \Delta t + o(\Delta t)$ . We write this as  $\Pr\{\text{arrival occurs between } t \text{ and } t + \Delta t\} = \lambda \Delta t + o(\Delta t)$ , where  $\lambda$  is a constant independent of  $t$ .
- $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$
- $\Pr\{\text{more than one arrival between } t \text{ and } t + \Delta t\} = o(\Delta t)$ ;

$N(t)$ ,  $\Delta t$  (is an incremental element, and  $o(\Delta t)$  denotes a quantity that becomes negligible when compared to  $\Delta t$  as  $\Delta t \rightarrow 0$ ; that is,

$$\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$$

- $\Pr\{\text{more than one arrival between } t \text{ and } t + \Delta t\} = o(\Delta t)$ ;

The numbers of arrivals in non-overlapping intervals are statistically independent; that is, the process has independent increments.

If we want to calculate  $p_n(t)$ , the probability of  $n$  arrivals in a time interval of length  $t$ ,  $n$  being an integer  $\geq 0$  then We will do this by first developing differential-difference equations for the arrival process as follows

$$p_n(t + \Delta t) = \Pr\{n \text{ arrivals in } t \text{ and zero in } \Delta t\}$$

$$+ \Pr\{n + 1 \text{ arrivals in } t \text{ and one in } \Delta t\}$$

$$\begin{aligned} & \Pr \{n \geq 2 \text{ arrivals in } t \text{ and two in } \Delta t\} \\ & \Pr \{0 \text{ arrivals in } t \text{ and } n \text{ in } \Delta t\} \quad (n \geq 1) \end{aligned} \quad (5.5)$$

Using assumptions (i), (ii) and (iii), Equation (5.5) becomes

$$p_n(t + \Delta t) = p_n(t)[1 - \lambda \Delta t - o(\Delta t)] + p_{n-1}(t)[\lambda \Delta t + o(\Delta t)] + o(\Delta t), \quad (5.6)$$

Where the last term,  $o(\Delta t)$ , represents the terms

$$\Pr \{n - j \text{ arrivals in } t \text{ and } j \text{ in } \Delta t; 2 \leq j \leq n\}.$$

For the case  $n = 0$ , we have

$$p_0(t + \Delta t) = p_0(t)[1 - \lambda \Delta t - o(\Delta t)]. \quad (5.7)$$

Rewriting (5.6) and (5.7) and combining all  $o(\Delta t)$  terms

$$p_0(t + \Delta t) - p_0(t) = -\lambda \Delta t p_0(t) + o(\Delta t) \quad (5.8)$$

and

$$p_n(t + \Delta t) - p_n(t) = -\lambda \Delta t p_n(t) + \lambda \Delta t p_{n-1}(t) + o(\Delta t) \quad (n \geq 1) \quad (5.9)$$

We divide (5.8) and (5.9) by  $\Delta t$ , take the limit as  $\Delta t \rightarrow 0$ , and obtain the following differential-difference equations

$$\begin{cases} \lim_{\Delta t \rightarrow 0} \left[ \frac{p_0(t + \Delta t) - p_0(t)}{\Delta t} = -\lambda p_0(t) + \frac{o(\Delta t)}{\Delta t} \right] \\ \lim_{\Delta t \rightarrow 0} \left[ \frac{p_n(t + \Delta t) - p_n(t)}{\Delta t} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(\Delta t)}{\Delta t} \right] \end{cases} \quad n \geq 1,$$

Which reduce to

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \quad (5.10)$$

and

$$\frac{dp_n(t)}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t) \quad (n \geq 1). \quad (5.11)$$

We now have an infinite set of linear, first order, ordinary differential equations to solve. Recall that a linear, first order, differential equation of the form

$$\frac{dy(x)}{dx} + \Phi(x)y(x) = \Psi(x) \quad (5.12)$$

has the solution

$$y(x) = Ce^{-\int \phi(x) dx} + e^{-\int \phi(x) dx} \int e^{\int \phi(x) dx} \Psi(x) dx, \quad (5.13)$$

Where  $c$  is a constant to be determined by boundary conditions. We will, therefore, use (5.13) to sequentially solve the infinite set of equations given by (5.10) and (5.11), utilizing mathematical induction to arrive at the final answer. It is left as an exercise to show that applying (5.13) to (5.10) and (5.11) for  $n = 0, 1$  and  $2$ , and using the boundary condition  $p_n(0) = 0$  for  $n > 0$  and  $p_0(0) = 1$  yields

$$\begin{cases} p_0(t) = e^{-\lambda t} \\ p_1(t) = \lambda t e^{-\lambda t} \\ p_2(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t} \\ p_3(t) = \frac{(\lambda t)^3}{3!} e^{-\lambda t} \end{cases} \quad (5.14)$$

From (5.14) we conjecture the general formula to be

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad (n \geq 0) \quad (5.15)$$

We will prove this by induction. Using (5.11) for  $n + 1$ , we have

$$\frac{dp_{n+1}(t)}{dt} + \lambda p_{n+1}(t) = \lambda p_n(t).$$

Let  $\Phi(t) = \lambda$  and  $\Psi(t) = \lambda p_n(t) = \frac{\lambda(\lambda t)^n}{n!} e^{-\lambda t}$ ; hence, from (5.15), (5.12) and (5.13), we obtain

$$\begin{aligned} p_{n+1}(t) &= Ce^{-\int \lambda dt} + e^{-\int \lambda dt} \int e^{\int \lambda dt} \frac{\lambda^{n+1} t^n}{n!} e^{-\lambda t} dt \\ &= Ce^{-\lambda t} + e^{-\lambda t} \frac{\lambda^{n+1} t^{n+1}}{(n+1)n!} \end{aligned}$$

Use of the boundary condition  $p_{n+1}(0) = 0$  yields  $c = 0$  and gives

$$p_{n+1}(t) = \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t}$$

Thus assuming (5.15) holds for  $n$ , we have shown it also for  $n + 1$ . we have proven it holds for  $n = 0, 1, 2$  and  $3$  in (5.14); thus the proof by induction is complete. Equation (5.15) is the well-known formula for the Poisson probability distribution with mean  $\lambda t$ . Thus, if we consider the random variable defined as the number of arrivals to a system in a time  $t$ , this random variable has the Poisson distribution given by (5.15) with a mean of  $\lambda t$  arrivals, or a mean arrival rate

of  $\lambda$ .

Poisson processes have a number of interesting additional properties. One of the most important is the fact that the numbers of occurrences in intervals of equal width are identically distributed (stationary increments). In particular, for  $t > s$ ,  $N(t) - N(s)$  is identically distributed as  $N(t+h) - N(s+h)$ , with frequency function

$$p_n(t-s) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^n}{n!}.$$

This can easily be seen by the following argument. Since the Poisson has independent increments [assumption iii], there is no loss of generality if  $N(s)$  and  $N(s+h)$  are assumed to be zero. Then, if the Poisson derivation is carried out for both  $N(t)$  and  $N(t+h)$  under assumptions (i), (ii) and (iii), the foregoing formula results for each.

Now we will show that if the arrival process follows the Poisson distribution, an associated random variable defined as the time between successive arrivals (interarrival time) follows the exponential distribution. Let  $T$  be the random variable “time between successive arrivals”; then

$$\Pr\{T \geq t\} = \Pr\{\text{zero arrivals in times } t\} = p_0(t) = e^{-\lambda t}.$$

Letting  $A(t)$  represent the cumulative distribution function of  $T$ , we have

$$A(t) = \Pr(T \leq t) = 1 - e^{-\lambda t}$$

The density function,  $a(t)$ , then is given by

$$a(t) = \frac{dA(t)}{dt} = \lambda e^{-\lambda t}.$$

Thus  $T$  has the exponential distribution with mean  $1/\lambda$ . We would intuitively expect the *mean* between arrivals to be  $1/\lambda$  if the *mean* arrival rate is  $\lambda$ . Our analysis substantiates this. It can also be shown that if the interarrival times are independent and have the same exponential distribution, then the arrival rate follows the Poisson distribution, and a proof of this assertion follows accordingly.

To being, let the cumulative distribution function (CDF) of the arrival counting process,  $\Pr\{N(t) \leq n\}$ , be denoted by  $P_n(t)$ . Then it follows that

$$P_n(t) = \Pr\{N(t) = n\}$$

$$= P_n(t) - P_{n-1}(t)$$

But,

$$P_n(t) = \Pr\{(\text{sum of } n+1 \text{ interarrival times}) > t\}.$$

However, the sum of independent and identical exponential random variables has an Erlang distribution (which is a special type of gamma distribution); hence

$$P_n(t) = \int_t^\infty \frac{\lambda(\lambda x)^n}{n!} e^{-\lambda x} dx \quad (5.16)$$

the transformation of variables  $u = x - t$  gives

$$\begin{aligned} P_n(t) &= \int_0^\infty \frac{\lambda^{n+1} (u+t)^n}{n!} e^{-\lambda t} e^{-\lambda u} du \\ &= \int_0^\infty \frac{\lambda^{n+1} e^{-\lambda t} e^{-\lambda u}}{n!} \sum_{i=0}^n u^{n-i} t^i \frac{n!}{(n-i)! i!} du, \end{aligned}$$

From the binomial theorem. The  $\sum$  and  $\int$  may be switched to give

$$P_n(t) = \sum_{i=0}^n \frac{\lambda^{n+1} e^{-\lambda t} t^i}{(n-i)! i!} \int_0^\infty e^{-\lambda u} u^{n-i} du.$$

But  $\int_0^\infty e^{-u} u^{n-i} du$  is the well-known gamma function denoted by  $\Gamma(n-i+1)$  which equals  $(n-i)!$  so with the proper change of variables to account for  $\lambda$  in the  $e^{-\lambda u}$  term of the integral, we get

$$P_n(t) = \sum_{i=0}^n \frac{(\lambda t)^i e^{-\lambda t}}{i!},$$

Which is clearly recognizable as the CDF of the Poisson process.

The Poisson/exponential arrivals process derived here is sometimes referred to as completely random arrivals. Although the reader might think that completely random would allude to some sort of haphazard arrival process or a uniform distribution for interarrival times, when encountered in queueing literature it specifically refers to the Poisson arrival rate-exponential interarrival time pattern. This can be explained in light of the following characteristic of a Poisson process. Given that  $k$  arrivals have occurred in an interval  $[0, T]$ , the  $k$  times  $\tau_1 < \tau_2 < \dots < \tau_k$  at which the arrivals occurred are distributed as the order statistics of  $k$  uniform random variables on  $[0, T]$ . Note that it is not interarrival times, but rather the times at which arrivals occurred, that are uniformly distributed. This can be proven as follows.

$$\begin{aligned} f_{\tau_1, \tau_2, \dots, \tau_k}(t_1, t_2, \dots, t_k | k \text{ arrivals in } [0, T]) dt_1 dt_2 \dots dt_k &\equiv f_\tau(t | k) dt \\ &= \Pr\{t_1 \leq \tau_1 \leq t_1 + dt_1, \dots, t_k \leq \tau_k \leq t_k + dt_k | k \text{ arrivals in } [0, T]\}. \end{aligned}$$

Using the definition of conditional probability gives

$$f_\tau(t | k) dt = \frac{\Pr\{t_1 \leq \tau_1 \leq t_1 + dt_1, \dots, t_k \leq \tau_k \leq t_k + dt_k \text{ and } k \text{ arrivals in } [0, T]\}}{\Pr\{k \text{ arrivals in } [0, T]\}}$$

The numerator of the last term above can be found by making direct use of the Poisson density function and its

properties, since we wish to find the probability that exactly one event occurs in each of the  $k$  time intervals,  $(t_i, t_i + dt_i)$ , and no events occur elsewhere, that is, in  $T - dt_1 - dt_2 - \dots - dt_k$ . Therefore, since the probability of  $k$  occurrences in a time  $t$  is  $(\lambda t)^k e^{-\lambda t} / k!$ , we have

$$f_\tau(t | k) dt = \frac{\lambda dt_1 e^{-\lambda dt_1} \lambda dt_2 e^{-\lambda dt_2} \dots \lambda dt_k e^{-\lambda dt_k} e^{-\lambda(T - dt_1 - dt_2 - \dots - dt_k)}}{(\lambda T)^k e^{-\lambda T} / k!},$$

Which reduces to?

$$f_\tau(t | k) dt = \frac{\lambda^k dt_1 dt_2 \dots dt_k e^{-\lambda T}}{(\lambda T)^k e^{-\lambda T} / k!} = \frac{k!}{T^k} dt_1 dt_2 \dots dt_k.$$

Hence

$$f_{\tau_1 \tau_2 \dots \tau_k}(t_1, t_2, \dots, t_k | k \text{ arrivals in } [0, T]) = \frac{k!}{T^k}, \quad (5.17)$$

Which is identical to the joint density of the order statistics of  $k$  random variables uniform on  $[0, T]$ .

Making similar assumptions as done above for arrivals, one could utilize the same type of process to describe the service pattern. If we change the three assumptions in the beginning of this section slightly by using the word service instead of arrival and condition the probability statements by requiring the system to be nonempty, we would obtain a Poisson service rate or an exponential service-time distribution for describing the service pattern. In the following section, we prove an important property of the exponential distribution which aids in a relatively simple analysis of queueing problems when arrival and service patterns exhibit the Poisson/exponential characteristics as derived in this section.

## CONCLUSIONS

These results unify and extend a number of separate results on networks of queues. The general model can have four types of service centers. Three of those types allow different service time distributions with rational Laplace transforms for different classes of customers. The model allows different classes of customers to have different arrival rates and different routing probabilities. For open networks some very simple formulas give the marginal distribution of customers at the service centers of the network.

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